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Interpolatory Subdivision Schemes and Wavelets*

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In this paper, we describe the connection of orthonormal wavelets to interpolatory subdivision. This connection leads us to a construction of an orthonormal wavelet of compact support whose Fourier transform has prescribed imaginary zeros. When all the zeros are at the origin, our construction reduces to the wavelet constructed by I. Daubechies.

1. INTRODUCTION

Recall that a *Stationary Subdivision Scheme* is an iterative method for the construction of curves (and surfaces). For planar curves, such a scheme begins with control points $\{\lambda_j : j \in \mathbb{Z}\}$ (vectors in $\mathbb{R}^2$) associated with the *coarse* lattice $\mathbb{Z}$. A rule for extending the control points to the *fine* lattice $\mathbb{Z}/2 = \{j/2 : j \in \mathbb{Z}\}$ is specified by a *mask* $\{a_j : j \in \mathbb{Z}\}$ of real numbers, always assumed to be of finite support so that the set

$$\{j : a_j \neq 0, j \in \mathbb{Z}\} \tag{1.1}$$

consists of a finite number of integers. The control points $\{\lambda_j^1 : j \in \mathbb{Z}\}$ at the next level are defined to be

$$\lambda_j^1 := (S\lambda)_j, \qquad j \in \mathbb{Z} \tag{1.2}$$

where

$$(S\lambda)_j := \sum_{k \in \mathbb{Z}} a_{j-2k} \lambda_k, \qquad j \in \mathbb{Z}. \tag{1.3}$$

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(Keep in mind that the control points $\tilde{\lambda}^1_{j/2} := \lambda^1_j$, $j \in \mathbb{Z}$ represent the new control points on the fine lattice $\mathbb{Z}/2$.) We view S as a linear operator on bounded sequences $\lambda = (\lambda_j)_{j \in \mathbb{Z}}$

$$\|\lambda\|_\infty = \sup\{|\lambda_j| : j \in \mathbb{Z}\} < \infty \tag{1.4}$$

and say that the stationary subdivision scheme S converges (in $\ell^\infty(\mathbb{Z})$) provided that for every bounded sequence λ there is an $f \in C(\mathbb{R})$ such that

$$\lim_{n \to \infty} \sup\{|(S^n\lambda)_j - f(j/2^n)| : j \in \mathbb{Z}\} = 0 \tag{1.5}$$

(f should be nonzero for some λ).

An essential feature of this iterative scheme for generating the curve f is that it is *stationary* and *homogeneous*. Its spatial invariance comes from noting that for any $e \in \{0, 1\}$

$$\tilde{\lambda}^1_{j+e/2} = \sum_{k \in \mathbb{Z}} a_{2j+e-2k} \lambda_k = \sum_{k \in \mathbb{Z}} a_{e+2k} \lambda_{j-k}, \qquad j \in \mathbb{Z}. \tag{1.6}$$

Specifically, these equations reveal that there are two rules which extend the initial control points $\{\lambda_j : j \in \mathbb{Z}\}$ to the fine lattice and each rule is independent of the "spatial" location $j \in \mathbb{Z}$. In other words, S consists of two Toeplitz matrices. At each further step of the iteration the *same* rules are used which insures the homogeneity of the iterative process.

Practical considerations sometimes demand that both nonstationary, nonhomogeneous and even nonlinear (in the initial control points) subdivision schemes be considered. Also, more complex coarse/fine lattice pairs $\mathbb{Z}$, $\mathbb{Z}/2$ are needed, especially for surface generation by subdivision. Elaborating on any of these important and interesting variations of the environment described above would get us far afield from our main purpose here. Information on these issues and many of the basic facts about stationary subdivision schemes can be found in [2], see also the papers [5–12, 16–19].

There is a special subclass of stationary subdivision schemes which are *interpolatory*. They have the property that the limit curve f in (1.5) interpolates the original control points, that is,

$$f(j) = \lambda_j, \qquad j \in \mathbb{Z}. \tag{1.7}$$

For instance, if at *each stage* of the iteration the previous control points are left unchanged then certainly (1.7) follows. According to (1.6) this demand on the subdivision scheme (1.3) is equivalent to the requirement on the mask that

$$a_{2j} = \delta_j, \qquad j \in \mathbb{Z}. \tag{1.8}$$

Equivalently, if we let $a(z)=\sum_{j\in\mathbb{Z}} a_j z^j$ then we have

$$a(z)+a(-z)=2, \qquad z\in\mathbb{C}\backslash\{0\}. \tag{1.9}$$

As far as we are aware, the earliest reference to interpolatory subdivision is the paper of Dubuc [8]. However, it is the recent beautiful paper of Deslauriers and Dubuc [6] which analyzes a class of interpolatory subdivision schemes that provides the motivation for this paper. We wish to elaborate upon the connection between this interesting work and the construction of compactly supported orthonormal wavelets by Daubechies in [3, 4] see also [1, 20]. In fact, many of the remarks we make here could have been written several years ago. Nonetheless, we feel it is beneficial to point out, especially within the geometric modelling community, this pleasant connection between orthonormal wavelets of Daubechies on one hand and the interpolatory subdivision scheme of Deslauriers and Dubuc on the other. Along the way, we will present some improvements and extensions of the ideas in [6] which may be useful for the further study of wavelets. These improvements will lead us to an extension of the orthonormal wavelet construction given in [3].

2. Some Background Material

Before we get to look at the interpolatory scheme from [6] we briefly review some basic general facts about interpolatory subdivision which follow from results in [2]. To this end, we set

$$a(z)=\sum_{j\in\mathbb{Z}} a_j z^j, \qquad z\in\mathbb{C}\backslash\{0\}.$$

Our first comment highlights the close connection between the convergence of interpolatory subdivision and the existence of a solution to a certain functional equation called *the refinement equation.*

Theorem 2.1. *Suppose the subdivision scheme* (1.3) *satisfies* (1.8) *and converges. Then*

$$a(-1)=0, \qquad a(1)=2 \tag{2.1}$$

and the limit f is given by

$$f(z)=\sum_{j\in\mathbb{Z}} \lambda_j \varphi(x-j), \qquad x\in\mathbb{R} \tag{2.2}$$

where φ is a continuous function of compact support which satisfies the refinement equation

$$\varphi(x)=\sum_{j\in\mathbb{Z}} a_j\varphi(2x-j), \qquad x\in\mathbb{R}, \tag{2.3}$$

and has the properties that

$$\sum_{j\in\mathbb{Z}} \varphi(x-j)=1, \qquad x\in\mathbb{R} \tag{2.4}$$

and

$$\varphi(j)=\delta_j, \qquad j\in\mathbb{Z}. \tag{2.5}$$

Conversely, if φ is a continuous solution of the refinement equation (2.3) *satisfying* (2.5) *then* (2.1) *holds and the subdivision scheme* (1.3) *converges to the function* (2.2).

This result describes the relationship between a continuous solution of the refinement equation (2.3) satisfying (2.5) and the convergence of (1.3).

Note that (2.1) means that $a(z)=(1+z)\,b(z)$ where

$$b(z)=\sum_{j\in\mathbb{Z}} b_j z^j, \qquad z\in\mathbb{C}\backslash\{0\} \tag{2.6}$$

satisfies $b(1)=1$. The order of zero of $a(z)$ at $z=-1$ regulates the (global) number of continuous derivatives that the limit of an interpolatory subdivision scheme possesses. This fact is described next.

THEOREM 2.2. *Suppose $\{a_j\}_{j\in\mathbb{Z}}$ is a finite mask which satisfies* (1.8) *and has a corresponding subdivision scheme which converges. Then the associated refinable function φ in* (2.3) *is in $C^r(\mathbb{R})$, $r\geqslant 1$, if and only if*

$$a(z)=\frac{(1+z)^{r+1}}{2^r}\,b(z)$$

for some $b(z)$ of the form (2.6) *such that $b(1)=1$ and the subdivision scheme corresponding to the mask $\{c_j\}_{j\in\mathbb{Z}}$ defined by*

$$\sum_{j\in\mathbb{Z}} c_j z^j := (1+z)\,b(z)$$

converges.

Another useful fact about the refinement equation (2.3) is the following special case of a result from [2], see also [14].

THEOREM 2.3. *Let φ be a continuous solution of compact support to the refinement equation*

$$\varphi(x)=\sum_{j\in\mathbb{Z}} a_j\varphi(2x-j), \qquad x\in\mathbb{R}.$$

Then

$$\sum_{j\in\mathbb{Z}} \varphi(x-j)=\int_{-\infty}^{\infty} \varphi(t)\,dt, \qquad x\in\mathbb{R}.$$

3. ORTHONORMAL WAVELETS OF DAUBECHIES AND SYMMETRIC ITERATIVE INTERPOLATION OF DESLAURIERS AND DUBUC

With these important general facts in mind, we now turn our attention to Deslauriers-Dubuc Interpolation. The interpolatory scheme from [6] begins with an integer $N\geqslant 1$ and then chooses the new control points $\{\tilde{\lambda}^1_{j+e/2}: j\in\mathbb{Z}\}$, $e\in\{0,1\}$ as follows: Fix an integer $j\in\mathbb{Z}$. Let p be the unique polynomial of degree $2N-1$ such that

$$p(j+\ell)=\lambda_{j+\ell}, \qquad \ell=-N+1, ..., N. \tag{3.1}$$

(Caveat: p depends on j.) Evaluate p at $j+e/2$ and set

$$\lambda^1_{2j+e}=\tilde{\lambda}^1_{j+e/2}:=p(j+e/2). \tag{3.2}$$

In particular, from (3.1) (for $\ell=0$) and (3.2) (with $e=0$) we see that

$$\tilde{\lambda}^1_j=\lambda_{2j}, \qquad j\in\mathbb{Z} \tag{3.3}$$

and so this is an interpolatory subdivision scheme. To identify its associated mask we let $\ell_j(x)$, $j=-N+1, ..., N$ be the Lagrange polynomials of degree $2N-1$ which have the property that

$$\ell_i(j)=\delta_{ij}, \qquad i,j=-N+1, ..., N \tag{3.4}$$

that is,

$$\ell_j(x)=(-1)^{N-1}\frac{\prod_{\ell=-N+1}^{N}(x-\ell)}{(N-1+j)!\,(N-j)!\,(x-j)}. \tag{3.5}$$

Then the polynomial p in (3.1) is given by

$$p(x) = \sum_{k=-N+1}^{N} \ell_k(x-j)\,\lambda_{j+1} \tag{3.6}$$

and hence

$$\tilde{\lambda}^1_{j+e/2} = \sum_{k=-N+1}^{N} \ell_k(e/2)\,\lambda_{j+k}. \tag{3.7}$$

Comparing this equation with (1.6) we see that

$$a_{e-2k} = \ell_k(e/2), \qquad k = -N+1, ..., N \tag{3.8}$$

(and zero otherwise).

Among other things it was proved in [6] that this subdivision scheme converges. The convergence proof was based on the observation that

$$a(e^{i\theta}) \geqslant 0, \qquad 0 \leqslant \theta \leqslant 2\pi \tag{3.9}$$

where equality holds *only at* $\theta = \pi$. This was proved in [6] by a Rolle's theorem argument. The proof below computes $a(e^{i\theta})$ directly and in particular confirms (3.9) by inspection

LEMMA 3.1.

$$\begin{aligned} a(e^{i\theta}) &= 2\,\frac{\int_\pi^\theta (\sin t)^{2N-1}\,dt}{\int_\pi^{2\pi} (\sin t)^{2N-1}\,dt} \\ &= 2\left\{1 - \frac{\int_0^\theta (\sin t)^{2N-1}\,dt}{\int_0^\pi (\sin t)^{2N-1}\,dt}\right\} \\ &= 2\left\{1 - \frac{(2N-1)!}{((N-1)!)^2\,2^{2N-1}} \int_0^\theta \sin^{2N-1} t\,dt\right\}. \end{aligned}$$

Proof. Note that the only possible nonzero values of a_j, $j \in \mathbb{Z}$ are $a_{-2N+1}, ..., a_{2N-1}$ and $a_{2\ell} = \delta_\ell$, $\ell = -N+1, ..., N-1$. Therefore we have in particular that

$$a(z) = \sum_{|j| \leqslant 2N-1} a_j z^j, \qquad z \in \mathbb{C}\backslash\{0\}. \tag{3.10}$$

Observe from definition (3.2) it follows that whenever $p \in \pi_{2N-1}$ and $\lambda_j := p(j)$, $j \in \mathbb{Z}$ then $\lambda^1_j = p(j/2)$, $j \in \mathbb{Z}$. We will make use of this property in the proof. To this end, we recall that generally the subdivision scheme S in

(1.3) has the property that whenever $\lambda_j := p(j)$, $j \in \mathbb{Z}$, where p is a polynomial of exact degree $m-1$ the sequence $(S\lambda)_j$ is likewise the value of a polynomial q at j, (that is, $q(j) = (S\lambda)_j$, $j \in \mathbb{Z}$, if and only if $a^{(j)}(-1) = 0$, $j = 0, 1, ..., m-1$). In other words, $a(z) = (1+z)^m r(z)$ for some Laurent polynomial $r(z)$, see [2]. Moreover, in this case, q is given by

$$q(x) = \frac{1}{2} \sum_{k \in \mathbb{Z}} a_k p\left(\frac{x-k}{2}\right), \qquad x \in \mathbb{R}. \tag{3.11}$$

If, in fact $q(x) = p(\frac{x}{2})$ as in the present situation then by (3.11)

$$p\left(\frac{x}{2}\right) = \frac{1}{2} \sum_{k \in \mathbb{Z}} a_k p\left(\frac{x-k}{2}\right), \qquad x \in \mathbb{R}. \tag{3.12}$$

Comparing the leading terms of both sides of (3.12) implies that

$$x^{m-1} = \frac{1}{2} \sum_{k \in \mathbb{Z}} a_k (x-k)^{m-1}, \qquad x \in \mathbb{R},$$

so that we have $a^{(j)}(1) = 0$, $j = 1, ..., m$. Applying this observation to the interpolatory subdivision scheme of Deslauriers and Dubuc we conclude that $a(-1) = 0$ and also

$$a^{(j)}(\pm 1) = 0, \qquad j = 1, ..., 2N-1.$$

Therefore, according to (3.10) there is a constant c such that

$$a'(z) = c(1+z)^{2N-1} (1-z)^{2N-1} z^{-2N}.$$

We let $z = e^{i\theta}$ and obtain

$$ia'(e^{i\theta})\, e^{i\theta} = c(\sin\theta)^{2N-1} (-1)^{N-1} 2^{2N-1}$$

from which it follows that

$$a(e^{i\theta}) = 2\, \frac{\int_\pi^\theta (\sin t)^{2N-1}\, dt}{\int_\pi^{2\pi} (\sin t)^{2N-1}\, dt}.$$

To evaluate the integrals

$$\rho_N := \int_0^\pi \sin^{2N-1} t\, dt$$

note that for $N > 1$

$$\begin{aligned}\rho_N &= \int_0^\pi \sin^{2N-1} t\, dt = \int_0^\pi (\sin^{2N-3} t)(1-\cos^2 t)\, dt \\ &= \rho_{N-1} - \int_0^\pi (\sin^{2N-3} t \cos^2 t)\, dt \\ &= P_{N-1} - \int_0^\pi \frac{d}{dt}\left(\frac{1}{2N-2} \sin^{2N-2} t\right) \cos t\, dt \\ &= \rho_{N-1} - \frac{1}{2N-2} \rho_N,\end{aligned}$$

from which it follows that

$$\begin{aligned}\rho_N &= \frac{2N-2}{2N-1} \rho_{N-1} = 2\left(\frac{N-1}{2N-1}\right) \rho_{N-1} \\ &= \cdots = \frac{2^{N-1}(N-1)!\, \rho_1}{(2N-1)(2N-3)\cdots 3} \\ &= \frac{2^{2N-1}((N-1)!)^2}{(2N-1)!}. \quad \blacksquare\end{aligned}$$

Let us now review the Daubechies compactly supported orthonormal smooth wavelet construction [3, 4]. The basic problem is to find a smooth function φ of compact support which satisfies a refinement equation of the form

$$\varphi(x) = \sum_{j \in \mathbb{Z}} b_j \varphi(2x - j), \qquad x \in \mathbb{R} \tag{3.13}$$

which has orthonormal integer translates. That is,

$$\int_{-\infty}^{\infty} \varphi(x)\, \varphi(x-j)\, dx = \delta_j, \qquad j \in \mathbb{Z}. \tag{3.14}$$

Once such a function φ has been identified then it follows that

$$\psi(x) := \sum_{j \in \mathbb{Z}} (-1)^j b_{1-j} \varphi(2x - j), \qquad x \in \mathbb{R} \tag{3.15}$$

has the property that the functions

$$\psi_{k,j}(x) = \sqrt{2^k}\, \psi(2^k x - j), \qquad j, j \in \mathbb{Z}$$

form an orthonormal basis of $L_2(\mathbb{R})$, [3, 4]. In other words, ψ is an orthonormal wavelet and it clearly has as many continuous derivatives as φ itself.

The relationship of the equations (3.12) and (3.13) to interpolatory subdivision is quite clear. We introduce the autocorrelation function of φ

$$F(x) := \int_{-\infty}^{\infty} \varphi(y)\, \varphi(y-x)\, dy$$

and observe that

$$F(x) := \sum_{j \in \mathbb{Z}} a_j F(2x-j)$$

where

$$\sum_{j \in \mathbb{Z}} a_j e^{ij\theta} := \frac{1}{2} \left| \sum_{j \in \mathbb{Z}} b_j e^{ij\theta} \right|^2 \tag{3.16}$$

and

$$F(j) = \delta_j, \qquad j \in \mathbb{Z}.$$

These formulas already suggest the relationship between the work of Deslauriers and Dubuc [6] and Daubechies [3]. To pin this down exactly, let us recall the details of the construction in [3] of a φ which satisfies both (3.13) and (3.14). First, we suppose M is chosen so that $b_j = 0$ for $j < 0$ or $j > M$, and consider the refinement equation in the form

$$\varphi(x) = \sum_{j=0}^{M} b_j \varphi(2x-j).$$

As a consequence of Theorem 2.3 any such function has the property that $\int_{-\infty}^{\infty} \varphi(x)\, dx = \pm 1$. Now, pick an N and try to find $a_0, ..., a_M$ so that not only do (3.13) and (3.14) hold but also ψ has N-vanishing moments. In other words,

$$\int_{-\infty}^{\infty} \psi(x)\, dx = \cdots = \int_{-\infty}^{\infty} \psi(x)\, x^{N-1}\, dx = 0.$$

A necessary and sufficient condition which insures that this is the case is that $b(-1) = \cdots = b^{(N-1)}(-1) = 0$. In fact, these conditions on b are equivalent to saying that

$$\sum_{j \in \mathbb{Z}} (-1)^j\, b_{1-j} q(j) = 0, \qquad q \in \pi_{N-1}$$

while for every $p \in \pi_{N-1}$

$$\int_{-\infty}^{\infty} \psi(x)\, p(x)\, dx = \sum_{j \in \mathbb{Z}} (-1)^j\, b_{1-j} r(j)$$

where r is the polynomial in π_{N-1} defined by

$$r(y) := \int_{-\infty}^{\infty} \varphi(2x-y)\, p(x)\, dx.$$

The mapping so defined from p to r is lower triangular and nonsingular since $\int_{-\infty}^{\infty} \varphi(x)\, dx \neq 0$. Therefore, indeed the moment conditions on ψ require that $b(z)$ has the form

$$b(z) = 2\left(\frac{1+z}{2}\right)^N S(z)$$

where S is a polynomial of degree at most $M-N$ with $S(1)=1$.

It is a simple fact that the orthogonality condition (3.14) implies that $b(z)$ satisfies the equation

$$|b(z)|^2 + |b(-z)|^2 = 4, \qquad |z| = 1. \tag{3.17}$$

Actually, using equation (3.16) in (1.9) proves (3.17). It was shown in [3, 4] that the set of all solutions to this algebraic equation has the form

$$|S(e^{i\theta})|^2 = P_N(\sin^2 \theta/2) + (\sin^{2N} \theta/2)\, R\left(\frac{1}{2}\cos\theta\right) \tag{3.18}$$

where R is any odd degree polynomial and

$$P_N(x) = \sum_{j=1}^{N-1} \binom{N-j-1}{j} x^j. \tag{3.19}$$

Hence the solution of least degree is

$$|S(e^{i\theta})|^2 = P_N(\sin^2 \theta/2). \tag{3.20}$$

In this case, $M = 2N-1$ and φ has support $(0, 2N-1)$. It was pointed out in [4, p. 978] that Meyer observed the formula

$$\left(\cos^2 \frac{1}{2}\theta\right)^N P_N\left(\sin^2 \frac{1}{2}\theta\right) = 1 - \frac{(2N-1)!}{[(N-1)!]^2\, 2^{2N-1}} \int_0^{\theta} \sin^{2N-1} t\, dt. \tag{3.21}$$

Hence

$$\begin{aligned}|b(e^{i\theta})|^2 &= 4(\cos^{2N}\theta/2)\,|S(e^{i\theta})|^2\\ &= 4(\cos^{2N}\theta/2)\,P_N\left(\sin^2\frac{1}{2}\theta\right)\\ &= 4\left\{1-\frac{(2N-1)!}{[(N-1)!]^2\,2^{2N-1}}\int_0^{\xi}\sin^{2N-1}x\,dx\right\}\\ &= 2a(e^{i\theta})\end{aligned}$$

i.e.

$$a(e^{i\theta}) = \frac{1}{2}\,|b(e^{i\theta})|^2.$$

This means that the Daubechies refinable function φ has support $(0, 2N-1)$ and its autocorrelation is the Lagrange function F of Deslauriers–Dubuc Interpolation. We shall extend this result in Section 4 and for this purpose we address the issue of convergence of interpolatory subdivision in the next section.

4. Convergence of Multivariate Interpolating Subdivision

In this section, we change the emphasis and present a simple sufficient condition for the convergence of *multivariate* interpolatory stationary subdivision schemes. Later we will apply this to a generalization of the interpolatory subdivision of Deslauriers and Dubuc and thereby be led to an extension of the Daubechies wavelet. Before we state our result, we review some standard multivariate notation. First, we use $|\cdot|_\infty$, $|\cdot|_1$ for the ℓ_∞ and ℓ_1 norm on $\mathbb{R}^s$, respectively. Also, $T=\{z=(z_1, ..., z_s)\colon z_i \in \mathbb{C}, |z_i| = 1, i = 1, ..., s\}$ is the s-torus and E stands for the extreme points of the s-cube, $[0, 1]^s$. We denote by $\hat{T}$ the subset of the s-torus consisting of all vectors of the form $(e^{ix_1}, ..., e^{ix_s})$ where $|x_i| \leqslant \pi/2$, $i = 1, 2, ..., s$. We also use $\rho := (1, ..., 1)^T$ for this special vector in E.

Motivated by our previous discussion we consider stationary subdivision on $\mathbb{R}^s$ of the form

$$(S\lambda)_i = \sum_{j\in\mathbb{Z}^s} a_{i-2j}\lambda_j, \qquad i\in\mathbb{Z}^s \tag{4.1}$$

where $\#\{j\colon a_j \neq 0, j\in\mathbb{Z}^s\} < \infty$ and say it is interpolatory provided that

$$a_{2i} = \delta_i, \qquad i\in\mathbb{Z}^s. \tag{4.2}$$

In this case, the symbol of $\{a_j\}_{j \in \mathbb{Z}^s}$

$$a(z) = \sum_{j \in \mathbb{Z}^s} a_j z^j,$$

is defined for $z = (z_1, ..., z_s)$ with $z \in (\mathbb{C} \backslash \{0\})^s$. Here we use the standard notation $z^j = z_1^{j_1} \cdots z_s^{j_s}$, $j = (j_1, ..., j_s)$, $z = (z_1, ..., z_s)$. Moreover, (4.2) is equivalent to the fact that $a(e) = 2^s$ and

$$\sum_{i \in E} a((-1)^i z) = 2^s, \qquad z \in (\mathbb{C} \backslash \{0\})^s. \tag{4.3}$$

THEOREM 4.1. *Suppose* $\{a_i\}_{i \in \mathbb{Z}^s}$ *is a finitely supported sequence with the properties that*

1. $a_{2i} = \delta_i, \qquad i \in \mathbb{Z}^s$ (4.4)
2. $a(z) \geqslant 0, \qquad z \in T$ (4.5)
3. *There exist vectors* $x^1, ..., x^n \in \mathbb{Z}^s$ *with* $|x^i|_1 = 1$, $i = 1, ..., n$ *and* a *finitely supported sequence* $\{b_i\}_{i \in \mathbb{Z}^s}$ *with* $b(z) \neq 0$, $z \in \hat{T}$, $b(\rho) = 1$ *such that*

$$a(z) = 2^{s-n} \prod_{i=1}^{n} (1 + z^{x^i})\, b(z). \tag{4.6}$$

Then there is a continuous refinable function φ *of compact support satisfying the refinement equation* (2.3) *such that* $\hat{\varphi} \in L_1(\mathbb{R}) \cap C(\mathbb{R})$, $\hat{\varphi}(w) \geqslant 0$, $w \in \mathbb{R}^s$ *and*

$$\varphi(j) = \delta_j, \qquad j \in \mathbb{Z}^s. \tag{4.7}$$

Moreover, the stationary subdivision scheme (4.1) *converges to* $f(x) = \sum_{i \in \mathbb{Z}^s} \lambda_i \varphi(x - i)$.

We remark that the condition on the vectors $x^1, ..., x^n$ implies that they are all coordinate directions. Therefore, in Theorem 4.1 we are assuming that

$$a(z) = 2^{s-n} \prod_{i=1}^{s} (1 + z_i)^{k_i}\, b(z), \qquad z = (z_1, ..., z_s)$$

where $k_1, ..., k_s$ are nonnegative integers such that $k_1 + \cdots + k_s = n$. Moreover, substituting $z = \rho$ into (4.3) and using (2) of Theorem 4.1 we conclude that each k_i, $i = 1, ..., s$ must be positive.

Proof. We define for $k = 1, 2, ...$ the Laurent polynomials

$$a^k(z) := a(z) \cdots a(z^{2^{k-1}\rho})$$
$$:= \sum_{j \in \mathbb{Z}^s} a_j^k z^j, \qquad z \in \mathbb{C} \backslash \{0\}.$$

Since $a^k(z), a(z)$ take a *vector* argument we interpret $z^{2^{k-1}\rho}$ as the vector $(z_1^{2^{k-1}}, ..., z_s^{2^{k-1}})$ where $z=(z_1, ..., z_s)$. Then it follows from (4.4) (by induction on k) that

$$a^k_{2^k j} = \delta_j, \qquad j \in \mathbb{Z}^s, \quad k=1, 2, ..., \tag{4.8}$$

and, in particular,

$$\frac{1}{(2\pi)^s} \int_{[-\pi, \pi]^s} a^k(e^{-ix})\, dx = 1, \qquad k=1, 2, \dots. \tag{4.9}$$

Again, as above, we interpret e^{-ix} as the vector $(e^{-ix_1}, ..., e^{-ix_s})$ where $x=(x_1, ..., x_s)$. Also, since $a(\rho)=2^s$ there is a constant $M>0$ such that

$$|1-2^{-s}a(e^{-ix})| \leqslant M\, |x|_\infty, \qquad x \in \mathbb{R}^s$$

and it follows that there is a function $\psi(x)$ continuous on $\mathbb{R}^s$ such that

$$\psi(x) = \lim_{k \to \infty} 2^{-ks} a^k(e^{-ix/2^k}). \tag{4.10}$$

Obviously, this convergence is uniform on compact sets and $\psi(x) \geqslant 0$, $x \in \mathbb{R}^s$. In fact, we will show that $\psi \in L_1(\mathbb{R}^s)$ and is strictly positive on $[-\pi, \pi]^s$. For this purpose, we observe that for $x \in \mathbb{R}^s$

$$\psi(x) = \prod_{\ell=1}^{\infty} \frac{1}{2^s} a(e^{-ix/2^\ell}) = \frac{1}{2^s} a(e^{-i(x/2)}) \prod_{\ell=1}^{\infty} \frac{1}{2^s} a(e^{-ix/2^{\ell+1}})$$

or

$$\psi(x) = \frac{1}{2^s} a(e^{-ix/2})\, \psi(x/2) \tag{4.11}$$

and for any $k \geqslant 1$

$$\psi(x) = 2^{-ks} a^k(e^{-ix/2^k})\, \psi(x/2^k). \tag{4.12}$$

Hence from (4.12), (4.5) and (4.9) we get

$$\begin{aligned} &\frac{1}{(2\pi)^s} \int_{[-2^k\pi, 2^k\pi]^s} \psi(x)\, dx \\ &= \frac{1}{(2\pi)^s} \int_{[-2^k\pi, 2^k\pi]^s} 2^{-ks} a^k(e^{-ix/2^k})\, \psi(x/2^k)\, dx \\ &= \frac{1}{(2\pi)^s} \int_{[-\pi, \pi]^s} a^k(e^{-ix})\, \psi(x)\, dx \\ &\leqslant \max\{\psi(x) \colon |x|_\infty \leqslant \pi\} \end{aligned}$$

which implies that $\psi \in L^1(\mathbb{R}^s)$.

Next, we provide a lower bound for ψ. To this end, we choose a positive γ such that $|b(z)| > \gamma$, for $z \in \hat{T}$. First, observe that

$$\prod_{\ell}^{\infty} = 1 \frac{1}{2^s} a(e^{-ix/2^\ell}) = \prod_{r=1}^{n} \prod_{\ell=1}^{\infty} \left(\frac{1+e^{-i(x^r \cdot x)/2^\ell}}{2} \right) \prod_{\ell=1}^{\infty} b(e^{-ix/2^\ell})$$

$$= \prod_{r=1}^{n} \left(\frac{1-e^{-ix^r \cdot x}}{ix^r \cdot x} \right) \prod_{\ell=1}^{\infty} b(e^{-ix/2^\ell}).$$

Now, choose a constant $c > 0$ such that

$$|1 - b(e^{ix})| \leqslant c\, |x|_\infty, \qquad x \in \mathbb{R}^s$$

and so it follows that whenever $\ell \geqslant \ell_0$, $|x|_\infty \leqslant \pi$ where $ec\pi \leqslant 2^{\ell_0}$,

$$|1 - b(e^{-ix/2^\ell})| \leqslant c\, 2^{-\ell}\, |x|_\infty \leqslant \pi c\, 2^{-\ell} \leqslant e^{-1}.$$

Therefore, since $1 - t \geqslant e^{-et}$, for $0 \leqslant t \leqslant e^{-1}$ we conclude that for $|x|_\infty \leqslant \pi$

$$\left| \prod_{\ell=1}^{\infty} b(e^{ix/2^\ell}) \right| \geqslant \gamma^{\ell_0 - 1} \prod_{\ell=\ell_0}^{\infty} (1 - |1 - b(e^{ix/2^\ell})|)$$

$$\geqslant \gamma^{\ell_0 - 1} \prod_{\ell=\ell_0}^{\infty} e^{-e\, |1 - b(e^{ix/2^\ell})|}$$

$$\geqslant \gamma^{\ell_0 - 1} \exp\left(-e \sum_{\ell=\ell_0}^{\infty} |1 - b(e^{ix/2^\ell})| \right)$$

$$\geqslant \gamma^{\ell_0 - 1} e^{-2^{1-\ell_0}\pi c e} \geqslant \gamma^{\ell_0 - 1} e^{-2} := \mu.$$

Consequently, we have for $|x|_\infty \leqslant \pi$ that

$$\psi(x) \geqslant \prod_{r=1}^{n} \left| \frac{\sin\, (x^r \cdot x)/2}{(x^r \cdot x)/2} \right| \mu$$

$$\geqslant \quad \mu \left(\frac{2}{\pi} \right)^n := \alpha.$$

Thus, when $|x|_\infty \leqslant 2^n \pi$ it follows from (4.12) that

$$\left| \frac{1}{2^s} a(e^{ix/2}) \cdots \frac{1}{2^s} a(e^{-ix/2^n}) \right|$$

$$\leqslant \rho^{-1} \left| \psi(x/2^n) \frac{1}{2^s} a(e^{-ix/2}) \cdots \frac{1}{2^s} a(e^{-ix/2^n}) \right|$$

$$= \alpha^{-1} \psi(x). \tag{4.13}$$

Moreover, from (4.8) we have that for $j \in \mathbb{Z}^s$

$$\delta_j = a^k_{2^k j} = \frac{1}{(2\pi)^s} \int_{[-\pi, \pi]^s} a(e^{-ix}) \cdots a(e^{-i2^{k-1}x})\, e^{i2^k j \cdot x}\, dx$$
$$= \frac{1}{(2\pi)^s} \int_{\mathbb{R}^s} \prod_{\ell=1}^{k-1} \frac{1}{2^s} a(e^{-ix/2^\ell})\, \chi_{[-\pi 2^k, \pi 2^k]^s}(x)\, e^{ij \cdot x}\, dx.$$

According to the bound (4.13) the integrand above is bounded by $\rho^{-1}\psi(x)$ and by the definition of ψ converges pointwise to $\psi(x)$. Thus by the Lebesgue dominated convergence theorem we conclude that

$$\delta_j = \frac{1}{(2\pi)^s} \int_{\mathbb{R}^s} \psi(x)\, e^{ij \cdot x}\, dx, \qquad j \in \mathbb{Z}^s. \tag{4.14}$$

Now, we define the continuous function

$$\varphi(y) := \frac{1}{(2\pi)^s} \int_{\mathbb{R}^s} \psi(x)\, e^{iy \cdot x}\, dx \tag{4.15}$$

so that

$$\varphi(j) = \delta_j, \qquad j \in \mathbb{Z}^s \tag{4.16}$$

and observe that since

$$\psi(x) = \frac{1}{2^s} a(e^{-ix/2})\, \psi(x/2), \qquad x \in \mathbb{R}^s \tag{4.17}$$

we have

$$\varphi(x) = \sum_{j \in \mathbb{Z}^s} a_j \varphi(2x - j), \qquad x \in \mathbb{R}^s. \tag{4.18}$$

It is easy to see that the function φ is of compact support. To this end, choose an integer $\rho > 0$ such that if $|j|_\infty > \rho$, $j \in \mathbb{Z}^s$ then $a_j = 0$. Now, we claim that $\varphi(j/2^k) = 0$ whenever $j \in \mathbb{Z}^s$ and $|j|_\infty > 2^k \rho$. We prove this by induction (the case $k = 0$ being taken care of by (4.16)). According to (4.18)

$$\varphi(j/2^k) = \sum_{|\ell|_\infty \leqslant \rho} a_\ell \varphi((j - \ell\, 2^{k-1})/2^{k-1}). \tag{4.19}$$

Since, for $|\ell|_\infty \leqslant \rho$ and $|j/2^k|_\infty > \rho$ it follows that

$$|j/2^{k-1} - \ell|_\infty \geqslant |j/2^{k-1}|_\infty - |\ell|_\infty$$
$$> 2\rho - \rho = \rho$$

we conclude by the induction hypothesis that indeed $\varphi(j/2^k)=0$ for any $j\in\mathbb{Z}^s$ with $|j/2^k|>\rho$. Using continuity of φ it follows that $\varphi(x)=0$, whenever $|x|_\infty>\rho$ and $x\in\mathbb{R}^s$.

We now can appeal to Theorem 2.3 and conclude that

$$\sum_{j\in\mathbb{Z}^s}\varphi(x-j)=\hat{\varphi}(0),\qquad x\in\mathbb{R}^s.$$

Thus by (4.16) we get that $\hat{\varphi}(0)=1$ so that (4.17) and (4.18) imply directly that $\hat{\varphi}=\psi$.

Finally, we observe that the subdivision (4.1) converges. This follows from general principles [2] but in this case is quite easy to see. In fact, let $\lambda\in\ell_\infty(\mathbb{Z}^s)$ then $f(x)=\sum_{j\in\mathbb{Z}^s}\lambda_j\varphi(x-j)$ is uniformly continuous on $\mathbb{R}^s$. Moreover, from the refinement equation (4.18) we see that

$$\begin{aligned} f(x)-\sum_{j\in\mathbb{Z}^s} f(j/2^k)\,\varphi(2^kx-j) \\ =\sum_{j\in\mathbb{Z}^s}((S^k\lambda)_j-f(j/2^k))\,\varphi(2^kx-j). \end{aligned}\tag{4.20}$$

Since φ is continuous of finite support and

$$\sum_{j\in\mathbb{Z}}\varphi(x-j)=1,\qquad x\in\mathbb{R}^s$$

the left hand side of (4.20) goes to zero as $k\to\infty$ uniformly for $x\in\mathbb{R}^s$. Hence, in view of the interpolation conditions (4.16) we conclude that

$$\lim_{k\to\infty}\sup\{|(S^k\lambda)_j-f(j/2^k)|: j\in\mathbb{Z}^s\}=0. \quad ∎$$

COROLLARY 4.1. *Suppose* $\{a_i\}_{i\in\mathbb{Z}}$ *is a finitely supported sequence such that*

$$a(z)+a(-z)=2,\qquad z\in\mathbb{C}\backslash\{0\}$$

$a(1)=2$ *and* $a(z)\geqslant 0$, *when* $|z|=1$ *with equality only if* $\operatorname{Re} z<0$. *Then the subdivision scheme* (1.3) *converges to a continuous refinable function* φ *of compact support such that* $\varphi(j)=\delta_j$, $j\in\mathbb{Z}$ *and* $\hat{\varphi}(x)\geqslant 0$, $x\in\mathbb{R}$.

We remark that this corollary, applied to the Deslauriers–Dubuc Interpolation, establishes that their iteration converges. In the general case, the last property of φ indicated in Corollary 4.1 suggests, as with the relationship between the autocorrelation of the Daubechies wavelet and the Lagrange fundamental function of Deslauriers Dubuc Interpolation that φ

is the autocorrelation of some function. In fact, we observe below that the φ of Corollary 4.1 can be expressed as the autocorrelation of a refinable function in $L^2(\mathbb{R})$ of compact support. Specifically, we have

COROLLARY 4.2. *Let $\{a_i\}_{i\in\mathbb{Z}}$ be a finitely supported sequence satisfying the hypothesis of Corollary 4.1 and φ the corresponding refinable function. Let b be any polynomial of degree N with real coefficients $\{b_j\}_{j=0}^N$ satisfying*

$$a(z)=\tfrac{1}{2}\,|b(z)|^2, \qquad |z|=1. \tag{4.21}$$

Then there is a refinable function $\theta\in L^2(\mathbb{R})$ with support in $(0, N)$ such that

$$\theta(x)=\sum_{j=0}^{N} b_j\theta(2x-j), \qquad a.e.\ x\in\mathbb{R} \tag{4.22}$$

and

$$\varphi(x)=\int_{\mathbb{R}} \theta(t)\,\theta(t-x)\,dt, \qquad x\in\mathbb{R}. \tag{4.23}$$

Proof. Since $a(z)\geqslant 0$ for $|z|=1$, Riesz's Lemma (cf. [4]) insures that there exists a polynomial b of degree N with real coefficients which satisfies (4.21) where N is the degree of $a(z)$, viz.

$$a(z)=\sum_{|j|\leqslant N} a_j z^j, \qquad z\in\mathbb{C}\backslash\{0\}.$$

For any such choice of b with $b(1)=2$ we let

$$H(x):=\prod_{\ell=1}^{\infty}\frac{1}{2}\,b(e^{-ix/2^\ell}), \qquad x\in\mathbb{R}. \tag{4.24}$$

Then H is a continuous function which satisfies the functional relation

$$H(x)=\tfrac{1}{2}\,b(e^{-ix/2})\,H(x/2), \qquad x\in\mathbb{R}. \tag{4.25}$$

Moreover, since

$$|H(x)|^2=\hat{\varphi}(x) \tag{4.26}$$

we conclude that $H\in L^2(\mathbb{R})$. Hence $H=\hat{\theta}$ for some $\theta\in L^2(\mathbb{R})$ and consequently by (4.25) it follows that

$$\theta(x)=\sum_{j=0}^{N} b_j\theta(2x-j) \qquad a.e.\ x\in\mathbb{R},$$

and by (4.26)

$$\varphi(x) = \int_{\mathbb{R}} \theta(t)\, \theta(t-x)\, dx, \qquad x \in \mathbb{R},$$

(both sides of this equation are continuous functions of x).

It remains to see that θ is of finite support. According to Lemma 6.2.2, p. 76 of [4] (attributed to Deslauriers and Dubuc) with $N_1 = 0$ and $N_2 = N$ we conclude that H is an entire function of exponential type N and therefore is the Fourier transform of a distribution with support on $[0, N]$. Since we have already pointed out that it is the Fourier transform of $\theta \in L^2(\mathbb{R})$ we conclude that θ has its support in $[0, N]$. ∎

5. Iterative Interpolation by Exponentials

In this section, we give an extension of the Deslauriers–Dubuc Interpolation and also of the Daubechies construction of orthonormal wavelets. We accomplish these extensions by studying the iteration described in the beginning of Section 3 using interpolation by *linear combinations of certain exponentials* rather than *polynomials*. To prepare for our analysis, we recall two facts from [13].

LEMMA 5.1. (Lemma 2.3, [13]). *Let the Laurent polynomial*

$$c(z) = \sum_{-n}^{n} c_j z^j$$

have only zeros in $(-\infty, 0)$ *where* c_j *are real constants with* $c_j = c_{-j}$, $|j| \geqslant n$. *Then there is a polynomial* p *of degree* n *such that*

$$p(x) = c(e^{i\omega}), \qquad x = \sin^2 \omega/2$$

and p *has only zeros in* $[1, \infty)$. *Moreover, if* $c(e^{i\omega}) \neq 0$, $|\omega| \leqslant \pi$ *then* $p(1) \neq 0$.

To apply this result we choose *any* n nonnegative numbers $x_1, ..., x_n$ such that

$$0 \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant 1. \tag{5.1}$$

We allow repetition in the numbers and with these nonnegative numbers consider the (symmetric) Laurent polynomial

$$d(z) = \prod_{j=1}^{n} \frac{(z + x_i)(z^{-1} + x_i)}{(1 + x_i)^2} \tag{5.2}$$

In the spirit of Lemma 5.1 we see that the polynomial

$$P(x) = \prod_{j=1}^{n} \left(1 - \frac{4x_i}{(1+x_i)^2}\, x\right), \tag{5.3}$$

has the property that

$$P(\sin^2 \omega/2) = d(e^{i\omega}), \qquad \omega \in \mathbb{R}. \tag{5.4}$$

To make use of this fact, we recall the following.

LEMMA 5.2 (Lemma 2.4, [13]). *Let p be a polynomial of degree n with all its zeros in $[1, \infty)$ having a leading coefficient of sign $(-1)^n$. Then there exists a unique polynomial q with real coefficients of degree $n-1$ such that*

$$q(x)\, p(x) + q(1-x)\, p(1-x) = 1,$$

and this polynomial has the property that

$$q(x) > 0 \qquad \textit{for} \quad x \in (0, 1).$$

Let us apply this fact to the polynomial P in (5.3) and obtain from P, via Lemma 5.2, a polynomial Q of degree $n-1$ such that

$$P(x)\ Q(x) + Q(1-x)\ P(1-x) = 1 \tag{5.6}$$

and

$$Q(x) > 0, \qquad x \in (0, 1). \tag{5.7}$$

We shall show that, in fact, $Q(0) \neq 0$. For this purpose, we note that

$$P(0) = 1, \qquad P(1) = \left(\prod_{j=1}^{n} \frac{1-x_i}{1+x_i}\right)^2. \tag{5.8}$$

To continue, we need to delve into the proof of Lemma 5.2 given in [13]. We make a distinction of cases. The easiest circumstance occurs when $1 \in \{x_1, ..., x_n\}$. In this case, equation (5.8) implies $P(0) = 1$ and $P(1) = 0$ and consequently choosing $x = 0$ in (5.6) we conclude $Q(0) = 1$. Next, suppose $1 \notin \{x_1, ..., x_n\}$, then $P(1) > 0$ by (5.8) and, in particular, P only has zeros in $(1, \infty)$. Let a, b represent the smallest, largest zero of P in this interval. In [13], it was shown in the proof of Lemma 5.2, as applied to the polynomials P and Q that the function PQ is positive and decreasing on the interval $(1-b, a)$. hence we conclude by this fact and equation (5.6) that

$$1 = q(1-b)\ P(1-b) > P(0)\ Q(0) > P(1)\ Q(1) > P(a)\ Q(a) = 0.$$

That is, since $P(0)=1$ we get

$$1>Q(0)>P(1)\,Q(1)>0$$

and in particular, $Q(0)\,Q(1)\neq 0$. Thus, in all cases, we have verified that $Q(0)\neq 0$ as claimed above.

We now define the Laurent polynomial

$$a(z):=2d(z)\,b(z), \qquad z\in\mathbb{C}\backslash\{0\} \tag{5.9}$$

where b is the symmetric Laurent polynomial of degree $n-1$ defined by

$$b(e^{i\omega}):=Q(\sin^2\omega/2), \qquad \omega\in\mathbb{R}. \tag{5.10}$$

We summarize in the next proposition several useful properties of the polynomial $a(z)$.

PROPOSITION 5.1. *For any $0<x_1\leqslant\cdots\leqslant x_n\leqslant 1$ there exists a unique Laurent polynomial*

$$a(z)=\sum_{|j|\leqslant 2n-1} a_j z^j, \tag{5.11}$$

with $a_j=a_{-j}=\bar{a}_j$, $|j|\leqslant 2n-1$ such that

$$a(z)+a(-z)=2, \qquad z\in\mathbb{C}\backslash\{0\} \tag{5.12}$$

and

$$a(-x_i)=a(-x_i^{-1})=0, \qquad i=1, ..., n \tag{5.13}$$

(where derivatives of $a(z)$ are taken for multiple x_i's). Moreover, this polynomial has the property that

$$a(z)\geqslant 0, \qquad |z|=1 \tag{5.14}$$

and equality occurs only if $z=-1$. When $1\in\{x_1, ..., x_n\}$ then equality holds if and only if $z=-1$.

Proof. The uniqueness of $a(z)$ satisfying the conditions of the proposition is clear. In fact, if a_1 and a_2 satisfy (5.11)–(5.13), then the difference $g=a_1-a_2$ has $4n$ zeros at $-x_i$, $-x_i^{-1}$, x_i, x_i^{-1}, $i=1, ..., n$ (counting multiplicities) and hence must be identically zero.

Our previous discussion provides the existence of $a(z)$ the Laurent polynomial. In fact, $a(z)$ given by (5.9) clearly vanishes at $-x_i$, $-x_i^{-1}$, $i=1, ..., n$ since $d(z)$ given by (5.2) has these values as zero. Moreover, $a(z)$ has the form (5.11) because both $d(z)$ and $b(z)$ are symmetric, have real coefficients and are of degree n and $n-1$, respectively. Finally, establishing equation (5.12) is equivalent to showing that

$$P(\sin^2 \omega/2)\, Q(\sin^2 \omega/2) + P(\sin^2(\omega-\pi)/2)\, Q(\sin^2(\omega-\pi)/2) = 1.$$

However, since $\sin^2(\omega-\pi)/2 = 1-\sin^2 \omega/2$ this is equivalent to equation (5.6) with $x = \sin^2\omega/2$.

Our last assertion follows from observing that

$$a(e^{i\omega}) = 2 \prod_{j=1}^{n} \frac{|e^{i\omega}+x_i|^2}{(1+x_i)^2}\, Q(\sin^2 \omega/2) \tag{5.15}$$

which is clearly nonnegative. Moreover, if $a(z^{i\omega_0}) = 0$ than either $Q(\sin^2 \omega_0/2) = 0$ or $1 \in \{x_1, ..., x_n\}$. If indeed $Q(\sin^2 \omega_0/2) = 0$ then $\omega_0 = \pm\pi$ since we have already shown above that $Q(x) > 0$ for $x \in [0, 1)$. If $Q(\sin^2 \omega_0/2) \neq 0$ then $1 \in \{x_1, ..., x_n\}$ and hence $\omega_0 = \pm\pi$. ∎

We are now ready to make use of this proposition for the analysis of the following iterative interpolation scheme. We start with a set $\Lambda = \{\lambda_1, ..., \lambda_N\}$ of nonnegative numbers (where we allow multiplicities). Next, we enlarge this set to $\Gamma := \Lambda \cup (-\Lambda)$ and consider the associate set of exponentials with frequencies in Γ. Specifically, we let

$$\Gamma = \{\gamma_1, ..., \gamma_{2N}\}$$

and suppose $\mu_1, ..., \mu_k$ are the *distinct* elements in Γ where μ_j occurs with multiplicity m_j, $j = 1, ..., k$ in Γ. Therefore $m_1 + m_2 + \cdots + m_k = 2N$. We define the linear space

$$T_\lambda = \operatorname{span}\{x^r e^{\mu_j x} \colon 0 \leqslant r \leqslant m_j - 1, j = 1, ..., k\},$$

and the vector $\lambda := (\lambda_1, ..., \lambda_N)^T$. Recall the fact that T_λ is a Chebyshev space. Thus there are Lagrange functions $\ell_j(\cdot \mid \lambda) \in T_\lambda$, $j = -N+1, ..., N$ such that

$$\ell_i(j \mid \lambda) = \delta_{ij} \qquad i, j = -N+1, ..., N.$$

We now define a subdivision mask $\{a_j(\lambda)\}_{j\in\mathbb{Z}}$, as in (3.8), by setting

$$a_{e-2k}(\lambda) := \ell_k(e/2 \mid \lambda), \qquad k = -N+1, ..., N+1, \quad e \in \{0, 1\}, \tag{5.16}$$

and zero otherwise. Since the space T_λ is invariant under the change $x \to 1-x$ of the independent variable of functions in T_λ we conclude that

$$\ell_{1-k}(1-x \mid \lambda) = \ell_k(x \mid \lambda), \qquad k = -N+1, ..., N$$

from which it follows that $a_j(\lambda) = a_{-j}(\lambda)$, $|j| \leqslant 2N-1$.

To prove convergence of the stationary subdivision scheme with this mask $\{a_k(\lambda)\}_{k\in\mathbb{Z}}$, we will apply Corollary 4.1. To this end, consider the Laurent polynomial

$$a(z \mid \lambda) = \sum_{|j| \leqslant 2N-1} a_j(\lambda)\, z^j \qquad z \in \mathbb{C}\backslash\{0\}, \tag{5.17}$$

which is symmetric by the above remark. In the next lemma, we list properties of $a(z \mid \lambda)$ that will be needed in our convergence analysis.

Proposition 5.2. *Suppose* $0 \in \{\lambda_1, ..., \lambda_N\}$ *and define* $x_i := e^{-\lambda_i/2}$, $i = 1, ..., N$ *then*

$$a(z \mid \lambda) + a(-z \mid \lambda) = 2, \qquad z \in \mathbb{C}\backslash\{0\}, \tag{5.18}$$

and

$$a(-x_i \mid \lambda) = a(-x_i^{-1} \mid \lambda) = 0, \qquad i = 1, ..., N, \tag{5.19}$$

(*where derivatives of* $a(z \mid \lambda)$ *are taken for multiple* x_i*'s*). *Moreover,* $a(z \mid \lambda) \geqslant 0$ *for* $|z| = 1$ *with equality if and only if* $z = -1$.

Proof. The proof of this result proceeds as in our discussion of Deslauriers–Dubuc Interpolation. Specifically, for every $p \in T_\lambda$ we have

$$p(i) = \sum_{r\in\mathbb{Z}} a_{2r}(\lambda)\, p(i-r), \qquad i \in \mathbb{Z}$$

and

$$p(i+1/2) = \sum_{r\in\mathbb{Z}} a_{2r+1}(\lambda)\, p(i-r), \qquad i \in \mathbb{Z}.$$

Hence for all $x \in \mathbb{R}$

$$\sum_{r\in\mathbb{Z}} a_{2r}(\lambda)\, p(x-r) = \sum_{r\in\mathbb{Z}} a_{2r+1}(\lambda)\, p(x-1/2-r).$$

If, for instance $\lambda_1, ..., \lambda_N$ are distinct and different from zero this equation implies that for $j = 1, ..., N$

$$\sum_{r\in\mathbb{Z}} a_{2r}(\lambda)\, e^{\pm\lambda_j(x-r)} - \sum_{r\in\mathbb{Z}} a_{2r+1}(\lambda)\, e^{\pm\lambda_j(x-1/2-4)} = 0$$

which implies that $a(-x_i \mid \lambda) = a(-x_i^{-1} \mid \lambda) = 0$, $i = 1, ..., N$. A similar argument applies for multiple λ_i's.

These remarks establish equations (5.19) the proof of (5.18) follows directly from the fact that $a_{2k}(\lambda) = \delta_k$, $k \in \mathbb{Z}$. Finally, the nonnegativity of

$a(z \mid \lambda)$ is a consequence of Proposition 5.1, since (5.18) and (5.19) identify $a(z \mid \lambda)$ with the polynomial in that proposition with $x_i = e^{\lambda_i/2}$, $i = 1, ..., N$.

The above result with Corollary 4.1 establishes the convergence of the interpolatory subdivision scheme

$$(S_\lambda c)_i := \sum_{j \in \mathbb{Z}} a_{i-2j}(\lambda)\, c_j, \qquad i \in \mathbb{Z} \tag{5.20}$$

which we state formally in the theorem below.

THEOREM 5.1. *For any set of real numbers* $\{\lambda_1, ..., \lambda_N\}$ *containing zero, the interpolatory subdivision scheme* (5.20) *with mask define by* (5.16) *converges. The refinable function* $\varphi(x \mid \lambda)$ *is continuous and zero outside of* $(-2N+1, 2N-1)$. *Moreover, it satisfies the refinement equation*

$$\varphi(x \mid \lambda) = \sum_{k \in \mathbb{Z}} a_k(\lambda)\, \varphi(2x - k \mid \lambda), \tag{5.21}$$

the interpolation conditions

$$\varphi(k \mid \lambda) = \delta_k, \qquad k \in \mathbb{Z} \tag{5.22}$$

and the equation

$$1 = \sum_{j \in \mathbb{Z}} \varphi(x - j \mid \lambda), \qquad x \in \mathbb{R}. \tag{5.23}$$

Next, we use Corollary 4.2 to introduce the function $\theta(x \mid \lambda)$ which is a refinable function whose autocorrelation is $\varphi(x \mid \lambda)$. To this end, we choose any real polynomial M_{N-1} of degree $N-1$ such that

$$|M_{N-1}(e^{i\omega})|^2 = Q(\sin^2 \omega/2)$$

and set

$$c(z \mid \lambda) := 2 \prod_{i=1}^{N} \left(\frac{z + x_i}{1 + x_i}\right) M_{N-1}(z).$$

THEOREM 5.2. *Suppose the hypothesis of Theorem* 5.1 *holds. Then there is a* $\theta(\cdot \mid \lambda) \in L^2(\mathbb{R})$ *with support in* $(0, 2N-1)$ *such that*

$$\theta(x \mid \lambda) = \sum_{j=0}^{2N-1} c_j(\lambda)\, \theta(2x - j \mid \lambda)$$

and

$$\int_{-\infty}^{\infty} \theta(x \mid \lambda)\, \theta(x-j \mid \lambda)\, dx = \delta_j, \qquad j \in \mathbb{Z}.$$

Moreover, the corresponding orthonormal wavelet

$$\psi(x \mid \lambda) = \sum_{j \in \mathbb{Z}} (-1)^j c_{1-j}(\lambda)\, \theta(2x-j \mid \lambda) \tag{5.24}$$

has the property that

$$\psi(i\lambda_j) = 0, \qquad j = 1, ..., N \tag{5.25}$$

(*with multiplicities*).

Proof. Corollary 4.2 yields all the assertions above except (5.25). For this fact, we note that by (5.24)

$$\psi(\omega) = -e^{-i\omega/2}\hat{\theta}(\omega/2)\, c(-e^{i\omega/2})$$

from which (5.25) follows since $x_i = e^{-\lambda_i/2}$ and $c(-x_i) = 0$, $i = 1, ..., N$. ▮

Remark 5.1. When $\lambda = 0$, $\psi(\cdot \mid 0)$ is the Daubechies wavelet which we discussed earlier.

6. Algebraic Characterization of Interpolatory Subdivision Schemes

In this section, we give an algebraic characterization of univariate interpolatory subdivision schemes whose symbol has a zero of prescribed order at -1. According to Theorem 2.2 the order of the zero regulates the number of continuous derivatives of the refinable function φ. Also, it determines the accuracy of the interpolatory subdivision. That is, the integer r such that whenever the vector $\lambda = (\lambda_j)_{j \in \mathbb{Z}}$ has the form $\lambda_j = p(j)$, $j \in \mathbb{Z}$ for some $p \in \pi_{r-1}$ then $S\lambda$ has the same form, $(S\lambda)_j = q(j)$, $j \in \mathbb{Z}$ for some $q \in \pi_{r-1}$ (see the proof of Lemma 3.1). On another occasion the computation presented next will be used to construct other convergent interpolatory subdivision schemes whose symbol has a prescribed zero at $z = -1$.

Thus we consider the solution of the equation

$$a(z) + a(-z) = 2, \qquad z \in \mathbb{C} \backslash \{0\} \tag{6.1}$$

where a is a Laurent polynomial of the form

$$a(z) = (1+z)^N q(z), \qquad q(1) = 2^{-N+1} \tag{6.2}$$

for some Laurent polynomial q. We express the Laurent polynomial q in the form

$$q(z) = z^{-m} F(z) \tag{6.3}$$

where m is a nonnegative integer and F is a polynomial. Substituting (6.2) and (6.3) into (6.1) gives the equation

$$(1+z)^N F(z) + (-1)^m (1-z)^N F(-z) = 2z^m. \tag{6.4}$$

Since the polynomials $(1+z)^N$ and $(1-z)^N$ have no common zeros and are both of degree N it follows that there is a unique solution of (6.4) of degree $N-1$, cf. [21]. We now proceed to find all solutions of (6.4) of degree $\geqslant N-1$. To this end, we let $\deg F = N+\ell$ for some $\ell \in \{-1, 0, 1, ...\}$. Note that $a(z)$ has the form

$$a(z) = \sum_{-m}^{2N+\ell-m} a_j z^j$$

for some constants $\{a_j\}_{-m}^{2N+\ell-m}$, and since $a_0 = 1$ we conclude that

$$0 \leqslant m \leqslant 2N + \ell.$$

We set $n := 2N+\ell$ and introduce constants $e_{k,m}$ $k, m = 0, 1, ..., n$ by the definition

$$z^m = \sum_{k=0}^{n} e_{k,m} (1+z)^k (1-z)^{n-k}, \qquad m = 0, 1, ..., n. \tag{6.5}$$

By considering the function

$$\begin{aligned}(y+z)^n &= \sum_{m=0}^{n} \binom{n}{m} y^{n-m} z^m \\ &= \sum_{k=0}^{n} \binom{n}{k} (1+z)^k (1-z)^{n-k} 2^{-n} (-1)^{n-k} (1+y)^k (1-y)^{n-k}\end{aligned}$$

we see that (6.5) is equivalent to the formula

$$(-1)^{n-k} 2^{-n} \binom{n}{k} (1+y)^k (1-y)^{n-k} = \sum_{m=0}^{n} \binom{n}{m} e_{k,m} y^{n-m}.$$

Replacing z by $-z$ in (6.5) we conclude that

$$(-1)^m e_{n-k,m} = e_{k,m}, \qquad k, m = 0, 1, ..., n. \tag{6.6}$$

These constants lead us to introduce two polynomials

$$G(z) := \sum_{j=0}^{N-1} e_{j,m}(1+z)^j (1-z)^{N-1-j}$$

and

$$H(z) := \sum_{j=0}^{\ell} e_{N+j,m}(1+z)^j (1-z)^{\ell-j}.$$

With these polynomials we are led to the following complete characterization of the solution to equations (6.4).

Proposition 6.1. *Suppose* $0 \leqslant m \leqslant n := 2N + \ell$, $\ell \in \{-1, 0, 1, ...\}$ *and* F *is a polynomial of degree* $\leqslant N + \ell$. *Then* F *satisfies* (6.4) *if and only if*

$$F(z) = (1-z)^N H(z) + 2(-1)^m (1+z)^{\ell+1} G(-z) + (1-z)^N v(z) \tag{6.7}$$

where v *is any polynomial of degree at most* ℓ *which satisfies the equation*

$$v(z) + (-1)^m v(-z) = 0. \tag{6.8}$$

For the proof we begin with some preliminary observations. (In the computations that follow we drop the second subscript on $e_{k,m}$ and denote this constant simply by e_k.) Note that

$$\deg G \leqslant N-1, \qquad \deg H \leqslant \ell$$

and also, since

$$H(-z) = \sum_{j=0}^{\ell} e_{N+j,m}(1-z)^j (1+z)^{\ell-j}$$

$$= \sum_{j=0}^{\ell} e_{\ell-j+N,m}(1+z)^j (1-z)^{\ell-j}$$

(6.6) implies that

$$= (-1)^m \sum_{j=0}^{\ell} e_{N+j,m}(1+z)^j (1-z)^{\ell-j}.$$

In other words, we have

$$H(-z) = (-1)^m H(z). \tag{6.9}$$

Next, we derive an equation relating the polynomials G and H. According to (6.5) we have

$$
\begin{aligned}
z^m &= \sum_{j=0}^{N-1} e_j (1+z)^j (1-z)^{n-j} + \sum_{j=N}^{N+\ell} e_j (1+z)^j (1-z)^{n-j} \\
&\quad + \sum_{j=N+\ell+1}^{2N+\ell} e_j (1+z)^j (1-z)^{n-j} \\
&= (1-z)^{N+\ell+1} G(z) + (1-z^2)^N H(z) \\
&\quad + \sum_{j=0}^{N-1} e_{N+\ell+1+j} (1+z)^{N+\ell+1+j} (1-z)^{N-1-j}
\end{aligned}
$$

and now we use (6.6) to conclude that

$$
\begin{aligned}
z^m &= (1-z)^{N+\ell+1} G(z) + (1-z^2)^N H(z) \\
&\quad + (-1)^m (1+z)^{N+\ell+1} G(-z). \qquad (6.10)
\end{aligned}
$$

Proof. For the proof we write F in the form

$$
F(z) = \sum_{j=0}^{N+\ell} c_j (1+z)^j (1+z)^{N+\ell-j}
$$

for some constants $c_0, ..., c_{N+\ell}$. Substituting this formula into (6.4) and using (6.5) we get

$$
\begin{aligned}
&(1+z)^N \sum_{j=0}^{N+\ell} c_j (1+z)^j (1-z)^{N+\ell-j} \\
&\qquad + (1-z)^N (-1)^m \sum_{j=0}^{N+\ell} c_j (1-z)^j (1+z)^{N+\ell-j} \\
&\quad = 2z^m = 2 \sum_{j=0}^{2N+\ell} e_j (1+z)^j (1-z)^{n-j}
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
&\sum_{r=N}^{2N+\ell} c_{r-N} (1+z)^r (1-z)^{2N+\ell-r} \\
&\qquad + (-1)^m \sum_{s=0}^{N+\ell} c_{N+\ell-s} (1+z)^s (1-z)^{2N+\ell-s} \\
&\quad = 2 \sum_{j=0}^{2N+\ell} e_j (1+z)^j (1-z)^{2N+\ell-j}.
\end{aligned}
$$

Identifying coefficients of the polynomials

$$(1+z)^j (1-z)^{2N+\ell-j}, \qquad 0 \leqslant j \leqslant 2N+\ell$$

we conclude that

$$2e_j = \begin{cases} (-1)^m c_{N+\ell-j}, & 0 \leqslant j \leqslant N-1 \\ (-1)^m c_{N+\ell-j} + c_{j-N}, & N \leqslant j \leqslant N+\ell \\ c_{j-N}, & N+\ell+1 \leqslant j \leqslant 2N+\ell. \end{cases}$$

In other words,

$$c_j + (-1)^m c_{\ell-j} = 2e_{N+j}, \qquad j = 0, 1, ..., \ell \tag{6.11}$$

$$c_j = 2e_{N+j}, \qquad j = \ell+1, ..., N+\ell \tag{6.12}$$

where here we used equation (6.6). Also, from this equation and equation (6.11) we see that

$$c_j = e_{N+j} + v_j, \qquad j = 0, 1, ..., \ell-1$$

where, as required by (6.10)

$$v(x) := \sum_{j=0}^{\ell} v_j (1+z)^j (1-z)^{\ell-j}$$

satisfies

$$v(z) + (-1)^m v(-z) = 0.$$

Hence we conclude that

$$\begin{aligned} F(z) &= \sum_{j=0}^{\ell} c_j (1+z)^j (1-z)^{N+\ell-j} + \sum_{j=\ell+1}^{N+\ell} c_j (1+z)^j (1-z)^{N+\ell-j} \\ &= (1-z)^N \sum_{j=0}^{\ell} \{e_{N+j} + v_j\} (1+z)^j (1-z)^{\ell-j} \\ &\quad + 2 \sum_{j=0}^{N-1} e_{N+\ell+j+1} (1+z)^{\ell+1+j} (1-z)^{N-1-j} \\ &= (1-z)^N \{H(z) + v(z)\} + 2(-1)^m (1+z)^{\ell+1} \\ &\quad \times \sum_{j=0}^{N-1} e_{N-1-j} (1+z)^j (1-z)^{N-1-j} \\ &= (1-z)^N H(z) + (1-z)^N v(z) + 2(-1)^m (1+z)^{\ell+1} G(-z). \end{aligned}$$

Finally, we confirm that any polynomial solution of the form (6.7) does indeed solve (6.4). To see this we replace z by $-z$ in (6.10), multiply both sides of the resulting equation by $(-1)^m$ and add the result to (6.10). This gives us

$$2z^m = (1-z^2)^N (H(z) + (-1)^m H(-z)) + 2(1-z)^{N+\ell+1} G(z) + 2(-1)^m (1+z)^{N+\ell+1} G(-z). \tag{6.13}$$

Next, we use (6.7) and observe that

$$\begin{aligned}(1+z)^N F(z) + (-1)^m (1-z)^N F(-z) \\ = (1-z^2)^N \{H(z) + (-1)^m H(-z)\} + (1-z^2)^N \{v(z) + (-1)^m v(z)\} \\ + 2(1-z)^{N+\ell+1} G(z) + 2(-1)^m (1+z)^{N+\ell+1} G(-z)\end{aligned}$$

and so by (6.8) and (6.13)

$$(1+z)^N F(z) + (-1)^m (1-z)^N F(-z) = 2z^m. \quad \blacksquare$$

By definition, a symmetric scheme is one for which the mask satisfies the requirement that

$$a_j = a_{-j}, \qquad j \in \mathbb{Z}.$$

That is, $a(z) = a(z^{-1})$, $z \in \mathbb{C} \backslash \{0\}$. Note again that $a(z)$ has the form

$$a(z) = \sum_{-m}^{2N+\ell-m} a_j z^j$$

so that we conclude for a symmetric scheme

$$\ell = 2(m-N)$$

and then in this case

$$\{j\colon a_j \neq 0\} \subseteq \{-m, ..., m\}.$$

Consequently, we have $N \leqslant m$ and so the symmetric interpolatory scheme with least support corresponds to the choice $\ell = 0$ and $m = N$. Therefore, according to Proposition 6.1, this scheme is given uniquely by

$$\begin{aligned}a(z) &= z^{-N}(1+z)^N \{(1-z)^N H(z) + 2(-1)^N (1+z) G(-z)\} \\ &= (z^{-1}-z)^N H(z) + 2(-1)^N z^{-N}(1+z)^{N+1} G(-z).\end{aligned}$$

When $N = m = 2M - 1$ is odd, equation (6.7) implies that $H(z) = 0$ (since for $\ell = 0$ H is a constant). Also (6.8) implies that

$$z^{2M-1} = (1-z)^{2M} G(z) - (1+z)^{2M} G(-z)$$

where $\deg G \leqslant 2M - 2$ and

$$a(z) = -2z^{-2M+1}(1+z)^{2M} G(-z).$$

This remark also leads us to the statement that the autocorrelation of the Daubechies wavelet is the Lagrange function of Deslauriers–Dubuc interpolation.

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